Solving Equations with e and $\ln x$

We know that the natural log function $\ln(x)$ is defined so that if $\ln(a) = b$ then $e^b = a$. The *common log* function $\log(x)$ has the property that if $\log(c) = d$ then $10^d = c$. It's possible to define a logarithmic function $\log_b(x)$ for any positive base b so that $\log_b(e) = f$ implies $b^f = e$. In practice, we rarely see bases other than 2, 10 and e.

Solve for y:

1. $\ln(y+1) + \ln(y-1) = 2x + \ln x$
2. $\log(y+1) = x^2 + \log(y-1)$
3. $2 \ln y = \ln(y+1) + x$

Solve for x (hint: put $u = e^x$, solve first for u):

4. $\dfrac{e^x + e^{-x}}{e^x - e^{-x}} = y$
5. $y = e^x + e^{-x}$

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1. $\ln(y+1) + \ln(y-1) = 2x + \ln x$

$\ln(y+1)(y-1) = 2x + \ln x$

$(y+1)(y-1) = e^{2x+\ln x}$

$y^2 - 1 = e^{2x} \cdot e^{\ln x}$

$y^2 - 1 = e^{2x} \cdot x$

$y^2 = xe^{2x} + 1$

$y = \pm\sqrt{xe^{2x}+1}$ $\quad y \geq 0,$

$y = \sqrt{xe^{2x}+1}$

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2. $\log(y+1) = x^2 + \log(y-1)$

$$\log \frac{y+1}{y-1} = x^2$$

$$\frac{y+1}{y-1} = 10^{x^2}$$

$$y+1 = 10^{x^2}y - 10^{x^2}$$

$$(10^{x^2}-1)y = 10^{x^2}+1$$

$$y = \frac{10^{x^2}+1}{10^{x^2}-1}$$

3. $2\ln y = \ln(y+1) + x$

$$\ln y^2 - \ln(y+1) = x$$

$$\ln \frac{y^2}{y+1} = x$$

$$\frac{y^2}{y+1} = e^x$$

$$y^2 = e^x y + e^x$$

$$y^2 - e^x y - e^x = 0$$

$$y = \frac{e^x \pm \sqrt{e^{2x} + 4e^x}}{2}$$

4. $$\frac{e^x + e^{-x}}{e^x - e^{-x}} = y$$

$$\frac{u + \frac{1}{u}}{u - \frac{1}{u}} = y$$

$$u + \frac{1}{u} = uy - \frac{1}{u}y$$

$$u^2 + 1 = u^2 y - y$$

$$u^2(y-1) = y + 1$$

$$u^2 = \frac{y+1}{y-1} \qquad u = \pm\sqrt{\frac{y+1}{y-1}}$$

$$e^x = \pm\sqrt{\frac{y+1}{y-1}}$$

$$x = \frac{1}{2}\ln\frac{y+1}{y-1}$$

5. $y = e^{x} + e^{-x}$

$$= u + \frac{1}{u}$$

$$uy = u^2 + 1$$

$$u^2 - uy + 1 = 0$$

$$u = \frac{y \pm \sqrt{y^2 - 4}}{2}$$

$$x = \ln\left(\frac{y \pm \sqrt{y^2 - 4}}{2}\right)$$